Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number | Candidate Number

Pearson Edexcel Level 3 GCE

Monday 15 May 2023

Afternoon (Time: 1 hour 40 minutes) | **Paper reference** | **8FM0/01**

Further Mathematics

Advanced Subsidiary

PAPER 1: Core Pure Mathematics

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 10 questions in this question paper. The total mark for this paper is 80.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

Pearson

1.

$$\begin{pmatrix} x & 9 \\ y & z \end{pmatrix} - 3\begin{pmatrix} z & y \\ z & y \end{pmatrix} = k\mathbf{I}$$

where x, y, z and k are constants.

Determine the value of x, the value of y and the value of z.

(4)

1. $\begin{bmatrix} x & 9 \\ y & z \end{bmatrix} - 3\begin{bmatrix} z & y \\ z & y \end{bmatrix} = k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 2×2 Identity Matrix

$\begin{bmatrix} x-3z & 9-3y \\ y-3z & z-3y \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

$x - 3z = k$

$9 - 3y = 0$
$y = 3$

$y - 3z = 0$
$y = 3z$

$z - 3y = k$

$y = 3 = 3z$
$z = 1$

$z - 3y = k$
$1 - 3(3) = k = -8$

$x - 3z = x - 3(1) = -8$
$x = -5$

$x = -5 \quad y = 3 \quad z = 1$

2. $\quad f(z) = z^3 + az^2 + bz + 175 \qquad$ where a and b are real constants

Given that $-3 + 4\text{i}$ is a root of the equation $f(z) = 0$

(a) determine the value of a and the value of b.

(4)

(b) Show all the roots of the equation $f(z) = 0$ on a single Argand diagram.

(2)

(c) Write down the roots of the equation $f(z + 2) = 0$

(1)

2.a) Given that $-3+4i$ is a root, because $f(z)$ is a polynomial with real coefficients, the complex conjugate $-3-4i$ would also be a root.

METHOD 1 (expand brackets & compare coefficients):

$\therefore f(z) = (z-(-3+4i))(z-(-3-4i))(z+r)$

3rd root must be real, because there is already a complex conjugate pair

$\therefore f(z) = (z^2+6z+25)(z+r)$

$= (z^3 + rz^2 + 6z^2 + 6rz + 25z + 25r)$

$= z^3 + (6+r)z^2 + (6r+25)z + 25r = z^3 + az^2 + bz + 175$

$25r = 175$

$r = 7$

$6+r = 6+7 = a$ $\qquad$ $6r+25 = 6(7)+25 = b$

$a = 13$ $\qquad$ $b = 67$

METHOD 2 (using product/sum rules for polynomials):

We know that $f(z)$ has roots $-3+4i, -3-4i, r$ ← real root

$az^3 + bz^2 + cz + d = 0$

$-\frac{d}{a} =$ product of all 3 roots $= (-3+4i)(-3-4i)(r) = -175$

Question 2 continued

$(25)r = -175$

$r = -7$

$\therefore$ 3 roots are $-3+4i, -3-4i, -7$

$z^3 + az^2 + bz + 175 = 0$

$-a = \alpha + \beta + \gamma$

$-a = -3+4i-3-4i-7 = -13$

$a = 13$

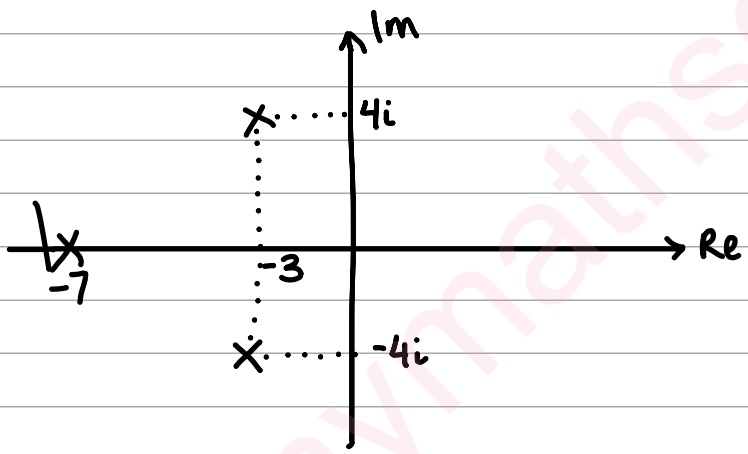
$b = \alpha\beta + \beta\gamma + \alpha\gamma$

$b = (-3+4i)(-3-4i) + (-3-4i)(-7) + (-3+4i)(-7)$

$b = 25 + 21 + 28i + 21 - 28i$

$b = 67$

b)



c) $f(z+2) = 0$

$f(z) = -2$ ← subtract 2 from each root of $f(z)$

$\therefore$ the roots are
$-5+4i, -5-4i, -9$

3. $$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

(a) Describe fully the single geometric transformation A represented by the matrix **A**.

(2)

$$\mathbf{B} = \begin{pmatrix} 1 & 3 & 0 \\ \sqrt{3} & 0 & 5\sqrt{3} \\ 1 & 2 & 0 \end{pmatrix}$$

The transformation B is represented by the matrix **B**.
The transformation A followed by the transformation B is the transformation C, which is represented by the matrix **C**.

To determine matrix **C**, a student attempts the following matrix multiplication.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 & 3 & 0 \\ \sqrt{3} & 0 & 5\sqrt{3} \\ 1 & 2 & 0 \end{pmatrix}$$

(b) State the error made by the student.

(1)

(c) Determine the correct matrix **C**.

(1)

3. a) Rotation of 30° about the x-axis

b) Because C is Matrix A followed by Matrix B

$\boxed{\text{Transformation X followed by Y is represented by the matrix YX}}$

$C = BA$ not AB

Question 3 continued

c) $\begin{bmatrix} 1 & 3 & 0 \\ \sqrt{3} & 0 & 5\sqrt{3} \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix}$

$= \begin{bmatrix} 1(1)+3(0)+0(0) & 1(0)+3(\sqrt{3}/2)+0(1/2) & 1(0)+3(-1/2)+0(\sqrt{3}/2) \\ \sqrt{3}(1)+0(0)+5\sqrt{3}(0) & \sqrt{3}(0)+0(\sqrt{3}/2)+5\sqrt{3}(1/2) & \sqrt{3}(0)+0(-1/2)+5\sqrt{3}(\sqrt{3}/2) \\ 1(1)+2(0)+0(0) & 1(0)+2(\sqrt{3}/2)+0(1/2) & 1(0)+2(-1/2)+0(\sqrt{3}/2) \end{bmatrix}$

$= \begin{bmatrix} 1 & 3\sqrt{3}/2 & -3/2 \\ \sqrt{3} & 5\sqrt{3}/2 & 15/2 \\ 1 & \sqrt{3} & -1 \end{bmatrix}$

(Total for Question 3 is 4 marks)

4. (i) (a) Show that

$$\frac{2+3\text{i}}{5+\text{i}} = k(1+\text{i})$$

where k is a constant to be determined.

(*Solutions relying on calculator technology are not acceptable.*)

(3)

Given that

- n is a positive integer
- $\left(\frac{2+3\text{i}}{5+\text{i}}\right)^n$ is a real number

(b) use the answer to part (a) to write down the smallest possible value of n.

(1)

(ii) The complex number $z = a + b\text{i}$ where a and b are real constants.

Given that

- $|z^{10}| = 59\,049$
- $\arg(z^{10}) = -\frac{5\pi}{3}$

determine the value of a and the value of b.

(4)

4.(i)a) $\frac{2+3i}{5+i} \times \frac{(5-i)}{(5-i)} = \frac{(2+3i)(5-i)}{(5+i)(5-i)}$

$= \frac{10-2i+15i-3i^2}{25+5i-5i-i^2}$

$= \frac{13+13i}{26} = \frac{1+i}{2} = \frac{1}{2}(1+i)$

$\therefore k = \frac{1}{2}$

b) $\left(\frac{2+3i}{5+i}\right)^n = \left(\frac{1+i}{2}\right)^n$

Question 4 continued

$\left(\frac{1+i}{2}\right)^2 = \frac{2i}{4}$ × not real

$\left(\frac{1+i}{2}\right)^3 = \frac{-2+2i}{8}$ × not real

$\left(\frac{1+i}{2}\right)^4 = \frac{-4}{16} = -\frac{1}{4}$ ✓ real

$\therefore n = 4$

(ii) $|z^{10}| = 59049$ $\quad |z^n| = |z|^n$

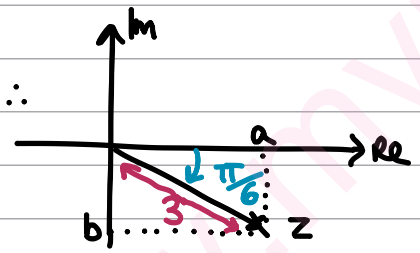
$\sqrt[10]{59049} = 3$ $\quad \arg(z^n) = n\arg(z)$

$\therefore |z| = 3$

$\arg(z^{10}) = 10\arg(z) = -\frac{5\pi}{3}$

$\arg(z) = -\frac{\pi}{6}$

$\therefore$



$a = 3\cos\frac{\pi}{6} = \frac{3\sqrt{3}}{2}$

$b = -\left(3\sin\frac{\pi}{6}\right) = -\frac{3}{2}$

$\therefore z = \frac{3\sqrt{3}}{2} - \frac{3}{2}i$

5. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

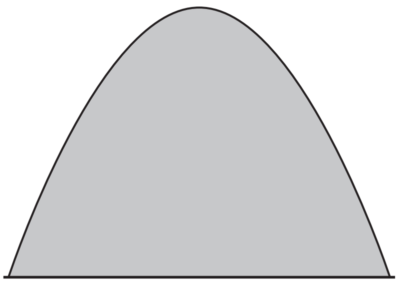


Figure 1

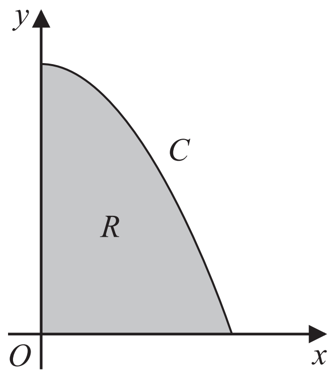


Figure 2

A large pile of concrete waste is created on a building site.

Figure 1 shows a central vertical cross-section of the concrete waste.

The curve C, shown in Figure 2, has equation

$$y + x^2 = 2 \qquad 0 \leqslant x \leqslant \sqrt{2}$$

The region R, shown shaded in Figure 2, is bounded by the y-axis, the x-axis and the curve C.

The volume of concrete waste is modelled by the volume of revolution formed when R is rotated through 360° about the y-axis. The units are metres.

The density of the concrete waste is $900\,\text{kg}\,\text{m}^{-3}$

(a) Use the model to estimate the mass of the concrete waste. Give your answer to 2 significant figures.
(6)

(b) Give a limitation of the model.
(1)

The mass of the concrete waste is approximately 5500 kg.

(c) Use this information and your answer to part (a) to evaluate the model, giving a reason for your answer.
(1)

5. a) Mass = Density × Volume

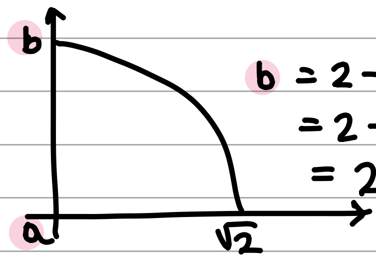
↑
900kgm^{-3}

$V_{\text{about y-axis}} = \pi \int_a^b x^2 \, dy$

Question 5 continued

$y + x^2 = 2$

$x = (2-y)^{\frac{1}{2}}$



$b = 2 - x^2$
$= 2 - (0)^2$
$= 2$

$V = \pi \int_0^2 \left((2-y)^{\frac{1}{2}}\right)^2 dy$

$= \pi \int_0^2 2 - y \; dy$

$= \pi \left[2y - \frac{y^2}{2} \right]_0^2$

$= \pi \left(2(2) - \frac{(2)^2}{2} \right) = 2\pi$

Mass $= 900 \times 2\pi$

$= 5654.9$ kg

$\therefore = 5700$ kg (2 s.f.)

b) • The surface won't be smooth (may have bumps)

• The pile may not exactly follow the shape of the curve

c) The difference between the model (5700 kg) & the actual mass (5500 kg) is very big (200 kg), so not a good model.

6. The line l_1 has equation $\mathbf{r} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ where λ is a scalar parameter.

The line l_2 is parallel to $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

(a) Show that l_1 and l_2 are perpendicular.

(2)

The plane Π contains the line l_1 and is perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

(b) Determine a Cartesian equation of Π

(2)

(c) Verify that the point $A(3, 1, 1)$ lies on Π

(1)

Given that

- the point of intersection of Π and l_2 has coordinates $(2, 3, 2)$
- the point $B(p, q, r)$ lies on l_2
- the distance AB is $2\sqrt{5}$
- p, q and r are positive integers

(d) determine the coordinates of B.

(6)

6. a) if vectors $a \cdot b = 0$, they're perpendicular

To show L_1 & L_2 are perpendicular, calculate dot product of the direction vectors of L_1 & L_2

$$\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 3(1) + 0(2) + 1(-3) = 0$$

point on the plane

b) Equation of a plane : $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

normal to the plane

The normal to the plane is $\begin{pmatrix}1\\2\\-3\end{pmatrix}$

because L_1 is on the plane, we can use the position vector from L_1

$$r \cdot \begin{pmatrix}1\\2\\-3\end{pmatrix} = \begin{pmatrix}-2\\2\\0\end{pmatrix} \cdot \begin{pmatrix}1\\2\\-3\end{pmatrix}$$

$$r \cdot \begin{pmatrix}1\\2\\-3\end{pmatrix} = (-2)(1) + (2)(2) + (0)(-3) = 2$$

$$x + 2y - 3z = 2$$

c) To check if A is on Π, plug into equation

$$\overset{r}{\begin{pmatrix}3\\1\\1\end{pmatrix}} \cdot \overset{n}{\begin{pmatrix}1\\2\\-3\end{pmatrix}} = (3)(1) + (1)(2) + (1)(-3) = 2$$

$\therefore$ A lies on Π

d) Equation of L_2 : $r = \begin{pmatrix}2\\3\\2\end{pmatrix} + \mu \begin{pmatrix}1\\2\\-3\end{pmatrix}$

$$\overset{B}{\begin{pmatrix}p\\q\\r\end{pmatrix}} = \begin{pmatrix}2\\3\\2\end{pmatrix} + \mu \begin{pmatrix}1\\2\\-3\end{pmatrix}$$

this is the point we're given lies on B $\therefore$ we can use it as a position vector

We are given that AB is $2\sqrt{5}$

$$|AB| = |\vec{OB} - \vec{OA}|$$

$$= \left| \begin{pmatrix}p\\q\\r\end{pmatrix} - \begin{pmatrix}3\\1\\1\end{pmatrix} \right|$$

← replace B by equation for L_2, using μ as a scalar parameter

Question 6 continued

$$\left|\begin{pmatrix}2+\mu\\3+2\mu\\2-3\mu\end{pmatrix}-\begin{pmatrix}3\\1\\1\end{pmatrix}\right| = \left|\begin{matrix}-1+\mu\\2+2\mu\\1-3\mu\end{matrix}\right| = 2\sqrt{5}$$

Modulus (i.e. distance) of a vector $\begin{pmatrix}a\\b\\c\end{pmatrix} = \sqrt{a^2+b^2+c^2}$

$$\sqrt{(-1+\mu)^2+(2+2\mu)^2+(1-3\mu)^2} = 2\sqrt{5}$$

$$1-2\mu+\mu^2+4+8\mu+4\mu^2+1-6\mu+9\mu^2 = 20$$

$$14\mu^2-14=0$$

$$\mu^2=1$$

$$\mu=\pm 1$$

$$\therefore B = \begin{pmatrix}2\\3\\2\end{pmatrix}+(1)\begin{pmatrix}1\\2\\-3\end{pmatrix}=\begin{pmatrix}3\\5\\-1\end{pmatrix} \text{ OR } \begin{pmatrix}2\\3\\2\end{pmatrix}+(-1)\begin{pmatrix}1\\2\\-3\end{pmatrix}=\begin{pmatrix}1\\1\\5\end{pmatrix}$$

Because we're given that p, q, r are positive integers

when $\mu=1$, r is negative $\therefore \mu=-1$

$$\therefore B=(1,1,5)$$

7. (i) Shade, on an Argand diagram, the set of points for which

$$|z - 3| \leqslant |z + 6\mathrm{i}|$$

(3)

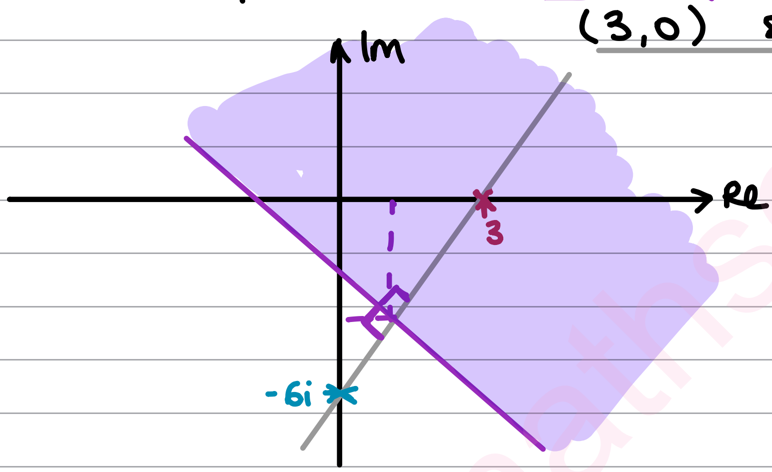
(ii) Determine the exact complex number w which satisfies both

$$\arg(w - 2) = \frac{\pi}{3} \quad \text{and} \quad \arg(w + 1) = \frac{\pi}{6}$$

(6)

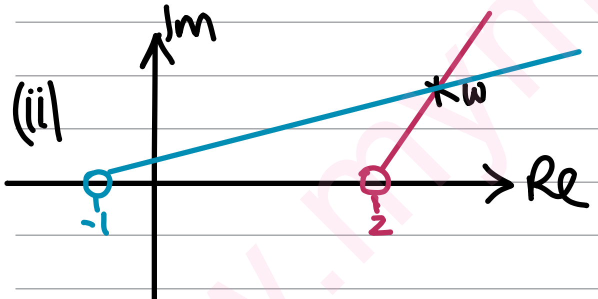
7. (i) $|z-3| \leq |z+6i|$ indicates a region on one side of the perpendicular bisector to the line between $(3,0)$ & $(0,-6)$

$|z-(3)| \leq |z-(-6i)|$



The inequality shows that the shaded area must be closer to $(3,0)$ than $(0,-6)$ $\therefore$ shade above the line.

(ii)



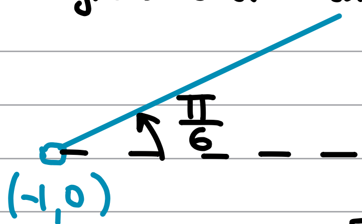
To find w, we can find the 2 equations of the lines & then calculate the point of intersection

To find equation of line: $(y-y_1=m(x-x_1))$

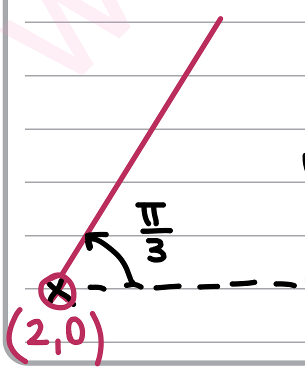
$\rightarrow$ gradient $= \frac{\Delta y}{\Delta x} = \frac{\text{opp.}}{\text{adj.}} = \tan\left(\frac{\pi}{3}\right)$

$= \sqrt{3}$

$\rightarrow$ gradient $= \tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$



$y-0=\sqrt{3}(x-2)$



$y=\sqrt{3}x-2\sqrt{3}$

$y-0=\frac{\sqrt{3}}{3}(x-(-1))$

$y=\frac{\sqrt{3}}{3}x+\frac{\sqrt{3}}{3}$

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Question 7 continued

$w \rightarrow \sqrt{3}x - 2\sqrt{3} = \frac{\sqrt{3}x}{3} + \frac{\sqrt{3}}{3}$

$x - 2 = \frac{x}{3} + \frac{1}{3}$

$\frac{2x}{3} = \frac{7}{3}$

$x = \frac{7}{2}$ $\quad y = \sqrt{3}\left(\frac{7}{2}\right) - 2\sqrt{3} = \frac{3\sqrt{3}}{2}$

$\therefore w = \frac{7}{2} + \frac{3\sqrt{3}}{2}i$

8. (a) Use the standard results for $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that, for all positive integers n,

$$\sum_{r=1}^{n}(2r-1)^2 = \frac{n}{3}(an^2-1)$$

where a is a constant to be determined.

(5)

(b) Hence determine the sum of the squares of all positive odd three-digit integers.

(3)

8.a) $\sum_{r=1}^{n}(2r-1)^2 = \sum_{r=1}^{n}(4r^2-4r+1)$

$= 4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$

$\boxed{\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1) \quad \sum_{r=1}^{n} n = \frac{1}{2}n(n+1) \quad \sum_{r=1}^{n} 1 = n}$

$= \frac{4}{6}n(n+1)(2n+1) - \frac{4}{2}n(n+1) + n$

$= \frac{n}{3}\left(2(n+1)(2n+1) - 6(n+1) + 3\right)$

$= \frac{n}{3}\left(2(2n^2+3n+1) - 6n - 6 + 3\right)$

$= \frac{n}{3}(4n^2+6n+2-6n-3)$

$= \frac{n}{3}(4n^2-1)$

b) sum of positive odd 3-digit integers

$= 101^2 + 103^2 + \ldots 997^2 + 999^2$

Using previous result for sum of $(\text{odd digits})^2 = (2r-1)^2$

Question 8 continued

$101 = 2(51) - 1 \quad \therefore \; r = 51$

$999 = 2(500) - 1 \quad \therefore \; r = 500$

$\therefore$ sum of (3 digit odd)^2

$$= \sum_{r=51}^{500} (2r-1)^2$$

split summation

e.g. $\displaystyle\sum_{r=1}^{500} r = \underbrace{1+2+\ldots 50}_{\sum_{r=1}^{50} r} + \overbrace{51+52+\ldots 499+500}^{\sum_{r=51}^{500} r}$

$$\sum_{r=51}^{500} (2r-1)^2 = \sum_{r=1}^{500} (2r-1)^2 - \sum_{r=1}^{50} (2r-1)^2$$

(USING FORMULA from (a))

$$= \frac{500}{3}\left(4(500)^2 - 1\right) - \frac{50}{3}\left(4(50)^2 - 1\right)$$

$$= 166666500 - 166650$$

$$= 166499850$$

(Total for Question 8 is 8 marks)

9. (i)

$$\mathbf{P} = \begin{pmatrix} k & -2 & 7 \\ -3 & -5 & 2 \\ k & k & 4 \end{pmatrix} \quad \text{where } k \text{ is a constant}$$

Show that **P** is non-singular for all real values of k.

(4)

(ii)

$$\mathbf{Q} = \begin{pmatrix} 2 & -1 \\ -3 & 0 \end{pmatrix}$$

The matrix **Q** represents a linear transformation T

Under T, the point $A(a, 2)$ and the point $B(4, -a)$, where a is a constant, are transformed to the points A' and B' respectively.

Given that the distance $A'B'$ is $\sqrt{58}$, determine the possible values of a.

(5)

9.(i) Singular matrices have a determinant of 0.

To show P is non-singular for all values of K, we must show the determinant of $P \neq 0$

3x3 DETERMINANT

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei-fh) - b(di-gf) + c(dh-eg)$$

$$|P| = K\begin{vmatrix} -5 & 2 \\ K & 4 \end{vmatrix} - (-2)\begin{vmatrix} -3 & 2 \\ K & 4 \end{vmatrix} + 7\begin{vmatrix} -3 & -5 \\ K & K \end{vmatrix}$$

$$|P| = K(-20-2K) + 2(-12-2K) + 7(2K)$$

$$= -20K - 2K^2 - 24 - 4K + 14K$$

$$= -2K^2 - 10K - 24$$

for $|P|$ to be singular : $-2K^2 - 10K - 24 = 0$

$$K^2 + 5K + 12 = 0$$

P 7 2 8 0 6 A 0 2 8 3 6

Question 9 continued

However looking at the discriminant (b^2-4ac)

$$(5)^2-4(1)(12) = -23 < 0$$

discriminant < 0

$\therefore$ no real solutions of k for which $|P| = 0$

$\therefore$ P is non-singular

(ii) $QA = A'$

$$\begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} a \\ 2 \end{bmatrix} = \begin{bmatrix} 2(a)-1(2) \\ -3(a)-0(2) \end{bmatrix} = \begin{bmatrix} 2a-2 \\ -3a \end{bmatrix}$$

$QB = B'$

$$\begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ -a \end{bmatrix} = \begin{bmatrix} 2(4)-1(-a) \\ -3(4)+0(-a) \end{bmatrix} = \begin{bmatrix} 8+a \\ -12 \end{bmatrix}$$

$A'B'$ = distance between $(2a-2, -3a)$ & $(8+a, -12)$

distance $= \sqrt{58} = \sqrt{(8+a-(2a-2))^2 + (-12-(-3a))^2}$

$$58 = (10-a)^2 + (-12+3a)^2$$

$$58 = 100-20a+a^2+144-72a+9a^2$$

$$10a^2-92a+186=0$$

$$a=3 \quad \cup \quad a=\frac{31}{5}$$

10. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

(i) The quartic equation

$$z^4 + 5z^2 - 30 = 0$$

has roots p, q, r and s.

Without solving the equation, determine the quartic equation whose roots are

$$(3p - 1), (3q - 1), (3r - 1) \text{ and } (3s - 1)$$

Give your answer in the form $w^4 + aw^3 + bw^2 + cw + d = 0$, where a, b, c and d are integers to be found.

(5)

(ii) The roots of the cubic equation

$$4x^3 + nx + 81 = 0 \qquad \text{where } n \text{ is a real constant}$$

are α, 2α and $\alpha - \beta$

Determine

(a) the values of the roots of the equation,

(5)

(b) the value of n.

(2)

10. (i) METHOD 1 (substitution):

$w = 3z - 1$ $\qquad$ $z^4 + 5z^2 - 30 = 0$

$z = \frac{w+1}{3}$ $\qquad$ $\left(\frac{w+1}{3}\right)^4 + 5\left(\frac{w+1}{3}\right)^2 - 30 = 0$

$\frac{w^4 + 4w^3 + 6w^2 + 4w + 1}{81} + \frac{5}{9}(w^2 + 2w + 1) - 30 = 0$

$w^4 + 4w^3 + 6w^2 + 4w + 1 + 45w^2 + 90w + 45 - 2430 = 0$

$w^4 + 4w^3 + 51w^2 + 94w - 2384 = 0$

Question 10 continued

METHOD 2 (using product/sum rules of polynomials):

$ax^4 + bx^3 + cx^2 + dx + e = 0$

sum of roots — sum of all possible products of 2 roots — $\alpha\beta\gamma\delta$

$\Sigma\alpha_i = -\frac{b}{a}$ $\quad \Sigma\alpha_i\alpha_j = \frac{c}{a}$ $\quad \Sigma\alpha_i\alpha_j\alpha_k = -\frac{d}{a}$ $\quad \Sigma\alpha_i\alpha_j\alpha_k\alpha_l = \frac{e}{a}$

sum of all possible products of 3 roots

$z^4 \_ + 5z^2 \_ - 30 = 0$

$p+q+r+s = 0$ $\quad pq+pr+ps+qr+qs+rs = 5$ $\quad pqr+pqs+prs+qrs = 0$ $\quad pqrs \quad -30$

For new equation: $(3p-1), (3q-1), (3r-1), (3s-1)$

$\Sigma\alpha_i = \frac{-b}{a} = 3p-1+3q-1+3r-1+3s-1 = 3(p+q+r+s) - 4$

$= 3(0) - 3 = -4 \qquad \therefore b = 4$

$\Sigma\alpha_i\alpha_j = \frac{c}{a} = (3p-1)(3q-1) + (3p-1)(3r-1) + (3p-1)(3s-1) +$
$(3q-1)(3r-1) + (3q-1)(3s-1) + (3r-1)(3s-1)$

$= 9(pq+pr+ps+qr+qs+rs) - 9(p+q+r+s) + 6$

$= 9(5) - 9(0) + 6 = 51 \qquad \therefore c = 51$

$\Sigma\alpha_i\alpha_j\alpha_k = -\frac{d}{a} = (3p-1)(3q-1)(3r-1) + (3p-1)(3q-1)(3s-1) +$
$(3p-1)(3p-1)(3r-1) + (3q-1)(3r-1)(3s-1)$

$= 27(pqr+pqs+prs+qrs) - 18(pq+pr+ps+qr+qs+rs)$

$+ 6(p+q+r+s) - 4$

Question 10 continued

$= 27(0) - 18(5) + 6(0) - 4$

$= -94 \qquad \therefore\ d = 94$

$\sum \alpha_i\alpha_j\alpha_k\alpha_l = \frac{e}{a} = (3p-1)(3q-1)(3r-1)(3s-1)$

$= 81(pqrs) - 27(pqr + pqs + prs + qrs) +$

$9(pq + pr + ps + qr + qs + rs) - 3(p+q+r+s) + 1$

$= 81(-30) - 27(0) + 9(5) - 3(0) + 1$

$= -2384 \qquad \therefore\ e = -2384$

$\therefore\ w^4 + 4w^3 + 51w^2 + 94w - 2384 = 0$

(ii) a) $4x^3 + nx + 81 = 0$

sum of roots $= -\frac{b}{a} = 0$

$\therefore\ \alpha + 2\alpha + \alpha - \beta = 0$

$4\alpha = \beta$

sum of pairs $= \frac{c}{a} = \frac{n}{4}$

$\therefore\ \alpha(2\alpha) + \alpha(\alpha - \beta) + 2\alpha(\alpha - \beta) = \frac{n}{4}$

$2\alpha^2 + \alpha^2 - \alpha\beta + 2\alpha^2 - 2\alpha\beta = \frac{n}{4}$

$5\alpha^2 - 3\alpha\beta = \frac{n}{4}$

$5\alpha^2 - 3\alpha(4\alpha) = \frac{n}{4}$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

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Question 10 continued

$$-7\alpha^2 = \frac{n}{4}$$

$$\alpha^2 = \frac{-n}{28}$$

$$\text{product} = -\frac{d}{a} = -\frac{81}{4}$$

$$\alpha(2\alpha)(\alpha-\beta) = -\frac{81}{4}$$

$$2\alpha^3 - 2\alpha^2\beta = -\frac{81}{4}$$ ← substituting from earlier

$$2\alpha^3 - 2\alpha^2(4\alpha) = \frac{-81}{4}$$

$$-6\alpha^3 = -\frac{81}{4}$$

$$\alpha^3 = \frac{27}{8}$$

$$\alpha = \frac{3}{2}$$

$\beta = 4\alpha = 6$

$\therefore$ roots $= \alpha, 2\alpha, \alpha-\beta$

$= 1.5, 3, -4.5$

b) as found in part (a)

$$\alpha^2 = -\frac{n}{28} \qquad \alpha = \frac{3}{2}$$

$$\therefore \frac{9}{4} = \frac{-n}{28}$$

$$n = -63$$